For the ellipse,

$$(2)\ \frac{dy}{dx}=-\frac{b^2x}{a^2y},\ \text{and}\ (3)\ \frac{d^2y}{dx^2}=-\frac{b^4}{a^2\,y^3}.$$

From (1), (2), and (3), making the sign positive,

$$\rho=\frac{(a^4y^2+b^4x^2)^{\frac{3}{2}}}{a^4b^4}.$$

Hence, for $(x_1,\ y_1)$, $\rho=R_1=\dfrac{(a^4y_1^2+b^4x_1^2)^{\frac{3}{2}}}{a^4b^4}$.

And for $-\dfrac{ay_1}{b},\ \dfrac{bx_1}{a},\ \rho=R_2=\dfrac{(a^2b^2x_1^2+a^2b^2y_1^2)^{\frac{3}{2}}}{a^4b^4}$.

$$\therefore\ R_1^{\frac{2}{3}}+R_2^{\frac{2}{3}}=\frac{(a^2+b^2)\ (b^2x_1^2+a^2y_1^2)}{(ab)^{\frac{8}{3}}}=\frac{a^2+b^2}{(ab)^{\frac{2}{3}}}.$$

MECHANICS.

235. Proposed by W. J. GREENSTREET, M. A., Marling School, Stroud, England.

A uniform heavy rod turns freely round a hinge at one end and rests with the other against a rough vertical wall, at angle, α, to the wall. Find the angle of arc on which this end may rest, and the pressures at the ends of the arc.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Let $2a$=length of rod; w, its weight; O, the hinge; OBC, the plane of the rod (vertical) perpendicular to the wall; α=angle BOC; μ=coefficient of friction between rod and wall; OA, the position of the rod at any time; OD, the projection of OA on the plane OBC; θ=angle DOA; ϕ=angle DOC; R= reaction or pressure at wall; OA, the axis of x; z, normal to AOD; y perpendicular to both x and z.

Then $OD=2a\cos\theta$, $OC=2a\sin\alpha$, $\phi=\cos^{-1}(\sin\alpha/\sin\theta)$.

The weight of the rod acting at its mid-point is equivalent to $w\cos\phi$ parallel to z, and $w\sin\phi$ parallel to OD. R is perpendicular to both OA and AD. Taking moments about y and z, respectively, we have, $aw\cos\phi=2aR$, $aw\sin\phi\sin\theta=2a\mu R$.

$\therefore\sin\theta=\mu\cot\theta=\mu\sin\alpha/\sqrt{(\cos^2\theta-\sin^2\alpha)}$.

$\therefore\sin^2\theta=\frac{1}{2}[\cos^2\alpha\pm\sqrt{(\cos^4\alpha-4\mu^2\sin^2\alpha)}]=P$. Use positive sign.

$R=(w/2)\cos\phi=[w\sin\alpha]/[2\sqrt{(1-P)}]$.

236. Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

A simple beam length $2a$, supported at the ends, is loaded with c pounds per running foot at the ends and increases uniformly to the center, where it is b pounds per running foot. Find deflection at center due to this load.

Solution by J. E. SANDERS, Weather Bureau, Chicago, Ill , and the PROPOSER.

The load is represented by a double trapezoid, bases b, c; altitude a for each. Let $AB=2a$, $AF=FB=a$, $AE=BC=c$, $DF=b$, G the place of any point distant $AG=x$ from the left end. Draw GH parallel to AE, EK parallel to AG. Then $EK=AG=x$, $GK=AE=c$, $HG=c+[(b-c)x]/a$.

Area $AGHE=x(2ac+bx-cx)/2a$, area $ABCDE=a(b+c)$, area $AGKE=cx$, area $EKH=(bx^2-cx^2)/2a$.

Let z=distance of center of gravity of $AGHE$ from GH.

Then $z(\text{area } AGHE)=\frac{1}{2}x(\text{area } AGKE)+\frac{1}{3}x(\text{area } EKH)$.

$\therefore zx(2ac+bx-cx)/2a=\frac{1}{2}cx^2+(bx^3-cx^3)/6a$.

$\therefore z=\dfrac{3acx+bx^2-cx^2}{3(2ac+bx-cx)}$.

Bending moment$=\frac{1}{2}a(b+c)x-z(\text{area } AGHE)$.

$\therefore EId^2y/dx^2=\frac{1}{2}a(b+c)x-\frac{1}{6}(3acx^2+bx^3-cx^3)/a$.

$EIdy/dx=\frac{1}{4}a(b+c)x^2-\frac{1}{6}cx^3-\dfrac{bx^4}{24a}+\dfrac{cx^4}{24a}+C$.

When $x=a$, $dy/dx=0$. $\therefore C=-\frac{5}{24}a^3b-\frac{1}{8}a^3c$.

$\therefore EIdy/dx=\frac{1}{4}abx^2+\frac{1}{4}acx^2-\frac{1}{6}cx^3-\dfrac{bx^4}{24a}+\dfrac{cx^4}{24a}-\frac{5}{24}a^3b-\frac{1}{8}a^3c$.

$\therefore EIy=\frac{1}{12}abx^2+\frac{1}{12}acx^2-\frac{1}{24}cx^4-\dfrac{bx^5}{120a}+\dfrac{cx^5}{120a}-\frac{5}{24}a^3bx-\frac{1}{8}a^3cx$.

No constant being added as x and y are zero together.

When $x=a$, $EIy=-\frac{2}{15}a^4b-\frac{3}{40}a^4c$. If $c=0$, $EIy=-\frac{2}{15}a^4b$. If $c=b$, $EIy=-\frac{5}{24}a^4b$.

If the load consisted of b pounds per running foot at the ends and decreased to c pounds per running foot at the middle, then z times area $AGHE=\frac{1}{2}x^2.HG+\frac{2}{3}x(\text{area } EKH)$ where $AE=b$ and K is on AE instead of GH as before; $HG=(ab-bx+cx)/a$; area $AGHK=(abx-bx^2+cx^2)/a$.

$z=\dfrac{3\,abx-bx^2+cx^2}{3\,(2ab-bx+cx)}$, the same as above by replacing c with b.

$\therefore$ Bending moment$=\frac{1}{2}a(b+c)x-(3abx^2-bx^3+cx^3)/6a$.

$\therefore EId^2y/dx^2=\frac{1}{2}a(b+c)x-(3abx^2-bx^3+cx^3)/6a$.

$EIdy/dx=\frac{1}{4}a(b+c)x^2-\frac{1}{6}(bx^3-\dfrac{bx^4}{4a}+\dfrac{cx^4}{4a})+C$.

When $x=a$, $dy/dx=0$; $\therefore C=-\frac{1}{8}a^3b-\frac{5}{24}a^3c$.

$\therefore EI\,dy/dx=\frac{1}{4}a(b+c)x^2-\frac{1}{6}bx^3+\dfrac{bx^4}{24a}-\dfrac{cx^4}{24a}-\frac{1}{8}a^3b-\frac{5}{24}a^3c$.

$EIy=\frac{1}{12}abx^3+\frac{1}{12}acx^3-\frac{1}{24}bx^4+\dfrac{bx^5}{120a}-\dfrac{cx^5}{120a}-\frac{1}{8}a^3bx-\frac{5}{24}a^3cx$.

No constant being added.

When $x=a$, $EIy=-\frac{3}{40}a^4b-\frac{2}{15}a^4c$.

If $c=0$, $EIy=-\frac{3}{40}a^4b$. If $b=c$, $EIy=-\frac{5}{24}a^4b$.

y is the deflection required in each case.

Also solved by S. G. Barton.